RIGID-ICE MODEL AND STATIONARY GROWTH OF ICE

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Abstract

A quasi-stationary version of the rigid-ice model that is a synthesis of models by Miller (1978) and Gilpin (1980) is analyzed. The model describes ice movement in a porous medium and its growth on the boundaries of the frozen fringe. The gradients of pressure and temperature are independent. Mass flows differ in the frozen fringe and the unfrozen zone. The solution of equations has one or three roots given stationary growth of ice. The solution with one root is stable. For the case with three roots, only the extreme two are stable. They differ from one another in their response to alternation of external temperature. The overburden pressure at which mass flow vanishes can always be found, but the temperature gradient in the frozen fringe and its length are not equal to zero. The calculated results were compared with data from modeling experiments.

Introduction

The complete rigid-ice model was presented by O’Neill and Miller (1985) for description of texture formation and frost heave at non-stationary processes. The quasi-stationary versions of this model allow phase transition either throughout the frozen fringe (Holden et al., 1985) or on its boundaries only (Gilpin, 1980; Gorelik and Kolunin, 1996). The second version is simpler, but nevertheless displays close quantitative results with the non-stationary model for a large range of internal and external parameters (Gorelik et al., 1997).

The simple model can be transformed to describe ice lens growth without penetration of ice into porous media. Ozawa and Kinosita (1989) investigated similar processes in experiments with thin porous membranes in conditions of all-round cooling. Moreover, this model allows important generalization of the introduction of anisotropic stresses in the porous ice (Kolunin, 1996).

Basic equations

The proposed set of equations describes the process of individual lens growth (Figure 1) and therefore does not contain the equations for defining the stresses in the soil’s components and the criterion of ice lens formation:

\[ j_f = \frac{P_{w2} - P_{w1}}{a / C_f + 1 / c_h} \]
\[ j_u = \frac{P_{w3} - P_{w2}}{C_u / b} \]
\[ j_f = -(1 - n)\rho_i v_{\xi1} / \rho_w \]
\[ (1 - n)j_u - j_f = n(1 - n)\Delta\rho_{wi}a / \rho_w \]
\[ a' = v_{\xi1} - v_{\xi2} \]

\[ \lambda_f(t_2 - t_1) / a - \mu_0(t_1 - t_0) = \kappa\rho_i(1 - n)v_{\xi1} \]
\[ \mu_0(t_1 - t_2) - \lambda_f(t_2 - t_1) / a = \kappa\rho_inv_{\xi2} \]
\[ P_i / \rho_i - P_w / \rho_w = -\kappa\alpha / T_0 \]
\[ j_f = -C_f \frac{\partial P_w}{\partial z} \]
\[ C_f^{-1} = a^{-1}\int_0^d C_f dz \]

Figure 1. Scheme of ice growth on porous body surface. 1 - ice, 2 - porous body, 3 - water.
where \( t \) - temperature °C; \( P_w, P_i \) - water and ice pressure relative to standard atmospheric pressure; \( j \) - flux of water; \( u_\xi \) - velocity of freezing boundary relatively ice; \( \lambda, \mu \) - coefficients of thermal conductivity and heat transfer; \( C, \overline{C} \) - local and average coefficients of hydraulic conductivity; \( n \) - porosity; \( \rho_w, \rho_i \) - water and ice densities; \( \Delta \rho_{wi} = \rho_w - \rho_i \); \( K \) - latent heat of water-ice phase transition. Indexes 0, 1, 2, 3 belong to isotherms \( t_0, t_1, t_2, t_3 \) accordingly; \( f \) - to frozen zone and frozen fringe; \( u \) - to unfrozen zone. Sign ’ is derivation by time.

The equations represent the following: (1) - (2) are the integral form of Darcy’s law for water flow into frozen fringe and unfrozen zone; (3) - equation of mass continuity on isotherms \( t_1 \); (4) - equation of joint deformations of ice lens and pore ice; (5) - kinematics correlation; (6) - (7) - heat balance on isotherms \( t_1 \) and \( t_2 \); (8) - generalized Clausius-Clapeyron equation; (9) - differential form of Darcy’s law; (10) - connection of local and average hydraulic conductivity coefficients. The last three equations all work in the frozen fringe. Boundary conditions are given by temperatures \( t_0, t_3 \) and pressure \( P_{w3} \). Ice pressure at isotherm \( t_1 \) is equal to overburden pressure \( \sigma \)

\[
P_i = \sigma \quad [11]
\]

The full hydraulic conductivity of the frozen fringe includes the hydraulic conductivity \( c_{hi} \) of the water film between ice lens and the contacting framework of porous body. Its contribution is essential if the value of \( a \) is not greater than particle diameter (primary heaving). The value of \( c_{hi} = \kappa h_1^3 / \eta \), where \( \kappa \) - a number of order 1; \( \eta \) - water viscosity; \( h_1 \) - a number of pores per unit cross-section of filter; \( h_1 \) - thickness of unfrozen water film near the ice lens.

The values of \( t_1 \) and \( a \) must be determined for each moment of time in the solution procedure.

It must be noted that the water film pressure \( P_{w} \) in the frozen fringe is implied as the pressure in bulk water connected by a channel with the given point of the pore water. The hydraulic equilibrium is maintained on the opposite ends of this channel. Such an understanding of the value of \( P_{w} \) corresponds to the works by Koopmans and Miller (1966) and O’Neill and Miller (1985). Gilpin (1980) uses gauge (full) pressure in pore water \( P_{wfr} \), which depends on the distance from the surface of solid particle \( h \) and is associated with pressure \( P_{w} \):

\[
P_{wh} = P_w + \Pi(h) \quad [12]
\]

where \( \Pi(h) \) - is a known function of disjoining pressure (Derjaguin and Churaev, 1985). If curvature of the ice-water surface equals zero, then the connection may be presented in another way:

\[
P_{wh} = \rho_i P_w / \rho_w - K \rho_i T / T_0 \quad [13]
\]

It may be shown that substitution of pressure \( P_{w} \) by means of equation (12) transforms the set (1) - (11) to Gilpin’s model. The reverse is also true. Therefore, the heat and mass transfer equations of Gilpin’s model and that presented here are equivalent.

The thermodynamic equilibrium of ice - unfrozen water gives the theoretical dependence of the water layer thickness versus temperature and pressure in the frozen fringe (Gilpin, 1980):

\[
\Pi(h) = -K \rho_i T / T_0 - \Delta \rho_{wi} P_w / \rho_w \quad [14]
\]

This relationship completes the fundamental equations (1) - (11) and makes it possible to find the hydraulic conductivities \( C_f \) and \( c_{hi} \) theoretically. This way is based on basic principles of mechanics and thermodynamics and may be verified on some simplest systems.

Nakano (1990) supposes that equations (8) and (14) must be added by empirical relationships between the unfrozen water content and the temperature: \( h = h(t) \). This correlation allows the value of \( h \) to be eliminated from equation (14) and to receive pressure \( P_{w} \) as a function of temperature \( P_{w} = P_w (t) \). The substitution of \( P_{w}(t) \) in equation (12) permits mass flow to be expressed through the temperature gradient. But empirical correlation \( h = h(t) \) is unnecessary because the initial set of equations is standing incorrect (number of equations greater than the number of unknowns). Consequently, the following relationship must be written for the discussed model

\[
j_f = -C_f \frac{\partial P_w}{\partial z} = -C_f \frac{\partial h}{\partial z} \quad [15]
\]

However some approaches use the empirical equation \( h = h(t) \) instead of equation (14). But in this case equation (15) is also required. This conclusion remains true if the curvature of ice - water surface is not equal to zero.
Taking into account equations (4), (7) we have the relation

\[ j_u \neq j_f \]  \hspace{1cm} [16]

Analogous reasons, based on capillary effects, lead to relations (15) and (16) for O’Neill - Miller’s model. Therefore some of the conclusions in Nakano (1990, 1997) relative to rigid-ice model, are incorrect.

## General properties of solutions

Let us consider a stationary process \( (\nu_1 = \nu_2 = \nu_3) \) with the availability of the frozen fringe. The basic equations become algebraic. The boundary temperatures answer the demands: \( t_0 < 0, t_3 > 0 \). The coefficients \( \mu_0 \) and \( \mu_3 \) may be transformed to the following form

\[ \mu_0 = \lambda_f / \lambda; \mu_3 = \lambda_u / (l-a), \]

where \( \lambda_f, \lambda_u \) - effective coefficients of thermal conductivity above the isotherm \( t_1 \) and below the isotherm \( t_2 \); \( H, l \) - lengths of media from the isotherm \( t_1 \) to isotherms \( t_0 \) and \( t_3 \). The following designations are used:

\[ \alpha = a/l; \beta = H/l; \Delta_{10} = \lambda_f (t_1 - t_0); \Delta_{21} = \lambda_f (t_2 - t_1); \]

\[ \Delta_{32} = \lambda_u (\Delta_3 - \Delta_2). \]

### Stationary Growth of Ice

Excepting from equations (1) and (2) \( P_{w2} \) with the help (4), (8) and (11), the flow \( j_f \) is given as a function of \( \alpha \) and \( t_1 \):

\[ j_f = \left( \frac{a l}{C_f} + \frac{(1-\alpha)l}{(1-n)C_u} + \frac{1}{c_h} \right)^{-1} \left( \frac{P_{w3} - P_w}{\sigma - \frac{\kappa \rho_u t_f}{T_0}} \right) \]

Combining equations (6), (7) with the help of (3) we receive the equation for the definition of the unknown temperature \( t_1 \):

\[ \Delta_{10} / \beta + \Delta_{32} / (1-\alpha) = \kappa \rho_u j_f / (1-n) \]  \hspace{1cm} [19]

where \( \alpha \) and \( j_f \) are determined by equations (17) and (18). We shall designate the left-hand part of the last equation by \( f_h (t_1) \) (function of the heat flow), and the right-hand part by \( f_m (t_1) \) (function of the mass flow). The possible relative arrangement of these graphs is shown in Figure 2.

At fixed temperatures \( t_0, t_3 \) and given coefficients \( \lambda_f, \lambda_u \) the function \( f_h \) has an immovable position. The behaviour of function \( f_m \) depends on the coefficients of hydraulic conductivity and on overburden pressure. If the value of \( \sigma \) increases, then curve \( f_m \) moves downwards and its maximum shifts to the left. The crossing points of the \( f_h \) and \( f_m \) graphs define the solution of equation (19) relative to the temperature \( t_1 \). All other unknown values may be expressed through the derived value of \( t_1 \). The graphs have either one or three crossing points (Figure 2).

Equations (6), (7), (3) and (5) allow for an expression for the change of the frozen fringe length at non-stationary process:

\[ \alpha' = k (f_h - f_m) \]  \hspace{1cm} [20]

where \( k > 0 \) is some coefficient. The functions \( f_h \) and \( f_m \) are determined by stationary equations (17) - (19) near to stationary points (crossing points of the graphs).

If equation (19) has three roots, then according to equation (20), the small deviations of temperature \( t_1 \) from any extreme root force the system to return to its initial state (Figure 2). Similar deviations near the middle root cause the system to move away from this root. Therefore, only the two extreme states are stable. A very important and interesting question is to define what cir-
cumstances disturb the stability of the process and transfer the system from one stationary state to another. The answer to this question remains beyond the scope of the present research. We note only that this transition may be similar to the ice growth process of primary heaving (Gorelik and Kolunin, 1993).

If equation (19) has unique root, then the stationary point is steady everywhere.

Analysis of the stable solution shows that increasing the overburden pressure $s$ tends to reduce mass flows and to make the frozen fringe longer, with all other parameters fixed. This statement does not depend on the number of roots and their position regarding the maximum location of the function $f_m$.

In contrast, the response of system to a change in external temperatures is determined by the location of the equation root relative to this maximum. Decreasing the temperature $t_0$ only increases the mass flow if the root is located to the right of the maximum of function $f_m$ and decreases the flow if the root is located to the left. The former was recorded by Nakano and Takeda (1994), and the latter by Brovka et al. (1990).

**MECHANICAL EQUILIBRIUM**

Because a value of $s$ is inserted in the mass flow function $f_m$, it may be significant to $s^*$ that the following relationships are true simultaneously (point $t_1 = t_1^*$, curve 4, Figure 2):

$$f_h = f_m = 0 \quad [21]$$

In this case, mass flows are absent in system and ice lens neither grows nor melts. The function $f_{hv}$, which is defined by the difference of incoming and outgoing heat fluxes, is equal to zero, but thermal gradients are still present in the system.

The overall length of the sample $L = l + H$ is fixed in freezing soil experiments as a rule. If it is accepted $\lambda_f l = \lambda_f$ then equations (21) and (17), and relationship of $f_h$ give

$$t_1^* = t_0 + H \left[ t_2 - t_0 + \frac{\lambda_f}{\lambda_f} (t_3 - t_2) / \lambda_f \right] / [22]$$

$$s^* = L \frac{\lambda_f (t_2 - t_0)}{\lambda_f (t_2 - t_0) + \lambda_u (t_3 - t_2)} \quad [23]$$

As the value of $t_1^*$ is replaced by means of (8) and (11) in the last equation, it may be shown that the value of $a^*$ increases with the growth of the equilibrium overburden pressure $s^*$.

Substitution in equation (8) of $P_l$, from (11) and $t_1^*$ from (22) gives the relation between overburden pressure $s^*$, boundary temperatures $t_0, t_3$ and depth of the ice lens $H$:

$$s^* = \frac{\rho_l P_w}{\rho_w T_0} - \frac{\kappa \rho_l}{T_0} H \left[ t_2 - t_0 + \frac{\lambda_u}{\lambda_f} (t_3 - t_2) \right] \quad [24]$$

Therefore overburden pressure $s^*$ is defined not only by cold boundary temperature $t_0$, but the heat boundary temperature $t_3$ and the ice lens disposition.

Measurement of the location of the last lens and the temperature on its surface are difficult, but external values of $t_0, t_3, P_w$ are controlled in experiments. If it is true that the equation $P_w = 0$, the ice lens has a small thickness and is located near freezing plate ($H/L << 1$) then equation (24) gives

$$s^* = -\kappa \rho_l t_0 / T_0 \quad [25]$$

This relationship approximates the data of Radd and Oertle (1973) in field of the great value of $s^*$. But their graph does not cross the origin of ordinates as is required by equation (25). The experimental curve gives $t_0 < 0 \degree C$ if $s^* = 0$. This fact is explained by last term of the sum (24) if $H \neq 0$.

When $P_w \neq 0$, equation (24) makes clear more significant deviations from the generalized Clausius-Clapeyron equation which were fixed by Takashi et al. (1981). The authors remark directly on the availability of the ice lenses within the sample and far from the cooling plate.

If overburden pressure $s$ exceeds a value defined by equation (24), then the mass flows in the system become negative, i.e., ice melts (curve 5, Figure 2). In this regime, graphs $f_h$ and $f_{hv}$ always intersect at one point.

**Analyses of experiments**

The experiments considered below, represent the simplest modeling systems. They were designed to study growth of ice in rigid porous media under stationary conditions.
Brovka et al. (1990) took a packet of capillaries with smooth walls as the working body. The cold plate temperature was measured on the contact of the ice column and porous medium. The pressures in both phases of water were atmospheric.

The water flow was recorded for variable temperatures of the cold surface. But the heating surface temperature was changed too so that the difference of temperatures across the porous body remained the same. The mass flow is a non-linear function of the temperature \( t_1 \) (Figure 3a).

The different dependence is observed with increasing the temperature of the heated side and fixing the value on the cold side (Figure 3b).

For a theoretical description of this experiment, the hydraulic conductivities of separate elements of this system must be defined. This is determined by the unfrozen water films into the frozen fringe. If the depth of the ice penetration greater 1 mm then the hydraulic conductivities both in the unfrozen zone of the filter and the water film on the lower interface of the ice lens are negligible.

Let us assume that the porous medium consists of parallel capillaries of fixed diameter. Water in the frozen fringe flows along capillaries through identical channels. Every channel is a thin layer of unfrozen water between the ice cylinder and capillary surfaces. The local conductivity coefficient of this medium is defined by

\[
C_f = \pi \gamma rh^3 / 6 \eta
\]

[26]

so the average conductivity coefficient of the frozen fringe may be found by means of equations (10), (26) and (14) and the linear temperature profile:

\[
\overline{C}_f = \frac{5\pi \gamma r}{6 \eta} \left( \frac{T_0 A}{k \rho_i} \right)^{3/2} \frac{t_2 - t_1}{(-t_1)^{5/2} - (-t_2)^{5/2}}
\]

[27]

where \( r \) - effective radius of separate capillary channel.

The function \( \Pi(h) \) was taken for calculations as

\[
\Pi(h) = \frac{A}{h^2}
\]

[28]

where \( A = 4.4 \times 10^{-12} \) N. This form follows from the theory of the disjoining pressure (Derjaguin et al., 1985) for a film thickness of less than 100 Å.

Figure 3. Dependence of mass flow (solid curve) and length of frozen fringe (broken curve) versus temperature: a) of cooling plate (1: \( t_3 - t_0 = 7.5^\circ C \), 2: \( t_3 = 7.2^\circ C \)), b) of heating plate (\( t_0 = -0.3^\circ C \)).

Figure 4. Scheme of ice growth and dependence of water flow versus cooling plate temperature.
The results of the theoretical explorations are presented in Figure 3. In addition to curve 1, related to the experimental condition, the mass flow was calculated as a function of the cooling plate temperature for an unchanging temperature of the heating plate (curve 2, Figure 3b). Both graphs have a similar behaviour. If the temperature of the cooling plate is constant then increasing the heating plate temperature increases the mass flows (Figure 3b). Moreover, the graphs show the frozen fringe size $a$ (Figure 3). Unfortunately this value was not measured in the experiment.

The experimental investigation is a rare one in which mass flows decreased with decreasing cooling plate temperatures.

Vignes and Dijkema (1974) grew ice in a macropore connected with a water reservoir using single split capillary. This is the sole principal difference from the experiment considered above. They believed that growing ice filled all the cross-section of the pore, and that an equilibrium (unfrozen) water film existed between the ice column and all the lower surface of the pore.

They observed increasing mass flows with decreases in cooling plate temperature. In contrast, the theoretical calculations give decreasing mass flows for regimes both with ice penetration in the capillary and in its absence. The cause of this failure is the low conductivity coefficient $c_i$ which is followed by the small value of $\gamma$ and the extensive area of water film flow on the macropore surface (on the butt-end of the capillary wall).

In order to eliminate this contradiction, we accept that the ice growth is realized as shown in Figure 4. The distance between the ice cap and the lower pore surface is much bigger than the thickness of unfrozen water film. Tearing off the ice cap is promoted by the paraffin oil in the pore, which was filled before testing. The oil wets the quartz surface very well and has a density that is approximately equal to the ice density. In the suggested scheme, the size of the ice column foot is commensurate to the capillary width and so the conductivity of the unfrozen water film may be neglected in comparison to the hydraulic conductivity of the split capillary.

With this assumption, the computation results show that the water flow in the system is proportional to external negative temperature and the experimental data is explained quantitatively (Figure 4).

Biermans et al. (1978) used glass filters as the working body. They applied all-round cooling with an environment temperature in the range -0.02 to -0.05°C. The pressure in the water reservoir was decreased to -0.7 atm relative to atmospheric pressure. The experimental results for measuring the water flow are shown by symbols on Figure 5.

For all-round cooling ($t_3 = t_0 < 0$) it may be proved that the stationary solution of equations (1) - (8) and (11) is absent as $\alpha > 0$. Therefore the ice grows on the filter surface without its penetration into pores. This remark was taken into account in a detailed analysis of this experiment (Gorelik and Kolumin, 1993). The calculated curves are shown by solid lines on Figure 5 and include parts for ice melting. The main contribution to the hydraulic conductivity comes from the water film between the growing ice and the upper surface of filter.

The remarkable part of these three experiments is that only one region at a time makes a contribution to the overall hydraulic conductivity: first test - unfrozen water film in the region of the ice penetration into porous medium; second - unfrozen zone of the working body; third - water film between ice and butt-end surface of the filter.

**Discussion and conclusions**

As Black (1995) rightly notes, the hydraulic conductivity of the frozen fringe greatly influences the calculated results and their conformity to the experimental data. A theoretical value of the conductivity is required to improve the coincidence of the calculated and observed relationships. In all probability, this circumstance is connected with tearing away the dissolved air from crystallization front, forming microbubbles in pores and jamming conductivity channels. The essential significance of the dissolved air was noted by Horiguchi and Miller (1980). A special procedure for
degassing the system was developed in order to get the undropped mass flows in the ice-sandwich permeame-
ter. However very few investigations of this phenome-
on are presented in the literature.

As a whole, it may be affirmed that the rigid-ice 
model is scientifically grounded to a high degree. But 
further verification of this and other analogous models 
will be possible if the hydraulic conductivity of the soils 
containing ice is measured more exactly in experiments.

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